ON THE ORIGIN OF CONVECTION

(O VOZNIKNOVENII KONVEKTSII)

PMM Vol.30, № 6, 1966, pp.1000-1005

V.I. IUDOVICH

(Rostov-on-Don)

(Received July 7, 1966)

The method of investigating bifurcations developed in [1 and 2] is applicable to many hydrodynamic problems. In the present paper it is applied to investigate the origin of convection in a horizontal fluid layer heated from below.

Secondary stationary flows are of particular interest in the convection problem since the loss of stability is associated with these flows: "the principle of the change in stability" is not only valid here but has been proved rigorously [3]. It has also been proved that secondary stationary flows are generated by branching off from the state of rest [4 and 5].

The problem under consideration is invariant relative to the group of motions of a horizontal plane.

The single solution invariant relative to this whole group is the rest solution. When this solution is unstable, it is natural to expect the occurrence of solutions invariant relative to some subgroup of the group of motions. If the mentioned subgroup is generated by a pair of translations (in perpendicular directions), we arrive at doubly-periodic solutions (Section 1), and if invariance relative to rotation through a certain angle is required in addition, we arrive at solutions of hexagonal type (Section 2). As is known, precisely these latter are realized in convection experiments [6]. Deductions on the existence of doubly-pertodic convection flows are elucidated in Theorem 1.1, and the existence of solutions of hexagonal convection type is asserted in Theorem 1.2. The applied method has slight connection with the boundary conditions. Only for definiteness is it assumed that the boundaries of the layer are solid walls on each of which the temperature is specified.

1. Convection in a horizontal layer. 1. On the formulation of the problem. Let a fluid be enclosed between two fixed, horizontal solid planes $z = 0, h$, on each of which the temperature is constant.

The convection equations then have the stationary solution

$$\mathbf{v}_0 = 0, \quad T_0 = cz + c_0, \quad p_0 = -\beta g\,(^1/_2\, cz^2 + c_0 z) + \text{const} \tag{1.1}$$

Here $x_1, x_2, x_3 = z$ are Cartesian coordinates; the z-axis is directed vertically downward. Seeking a new stationary solution $(\mathbf{v}', T', p')$ as

$$\mathbf{v}' = \mathbf{v}_0 + \mathbf{v}, \qquad T' = T_0 + T, \qquad p' = p_0 + p \tag{1.2}$$

we arrive at the following system of equations

$$\nu\Delta\mathbf{v} - \nabla p = (\mathbf{v}, \nabla)\mathbf{v} + \beta T g, \qquad \chi\Delta T - \mathbf{v}\cdot\nabla T = cv_3 \tag{1.3}$$

The boundary conditions on the solid walls are

$$\mathbf{v} = 0, \qquad T = 0 \qquad (z = 0, h) \tag{1.4}$$

Furthermore, we assume $\mathbf{v}$, T to be periodic in x_1, x_2 with periods $2\pi/k_1$ and $2\pi/k_2$, respectively (k_1, k_2 are wave numbers), and that the fluid layer as a whole cannot be displaced along the x_1, x_2 plane

$$\int_{-\pi/k_2}^{\pi/k_2}\int_0^h v_1 dx_2 dx_3 = \int_{-\pi/k_1}^{\pi/k_1}\int_0^h v_2 dx_1 dx_3 = 0 \tag{1.5}$$

Henceforth, we seek bifurcation values of the parameter c, the temperature gradient.

2. Fundamental functional spaces and operator equations. Let us introduce the following Hilbert spaces.

a) The space H_1 in which the set of smooth solenoidal vectors, periodic with periods $2\pi/k_i$ in x_1, x_2 and satisfying conditions (1.4), (1.5) is everywhere dense, and the scalar product is

$$(\mathbf{v}', \mathbf{v}'')_{H_1} = \int_\Omega \sum_{k=1}^{3} \frac{\partial \mathbf{v}'}{\partial x_k}\frac{\partial \mathbf{v}''}{\partial x_k}\, dx \tag{1.6}$$

Here Ω is a parallelepiped;

$$\{0 \leqslant x_3 = z \leqslant h, \ |x_1| \leqslant \pi/k_1, \ |x_2| \leqslant \pi/k_2\}.$$

b) The subspace $H_1{}^\circ$ of the space H_1 consisting of vectors satisfying the following evenness and oddness conditions:

$$\begin{aligned} v_1(x_1, x_2, z) &= -v_1(-x_1, x_2, z) = v_1(x_1, -x_2, z) \\ v_2(x_1, x_2, z) &= v_2(-x_1, x_2, z) = -v_2(x_1, -x_2, z) \\ v_3(x_1, x_2, z) &= v_3(-x_1, x_2, z) = v_3(x_1, -x_2, z) \end{aligned} \tag{1.7}$$

c) H_2 is the closure of the set of smooth functions defined in the layer $0 \le z \le h$, periodic with periods $2\pi/k_1$, $2\pi/k_2$ in x_1, x_2 and vanishing for $z = 0, h$, into the metric

$$(T', T'')_{H_2} = \int_\Omega \nabla T' \nabla T'' dx \tag{1.8}$$

d) The subspace $H_2{}^\circ$ of the space H_2 consisting of even functions in x_1 and x_2.

The problem (1.3) to (1.5) can be reduced to an operator equation with a completely continuous operator by many methods. For example, as has been shown in [5], the problem (1.3) to (1.5) is equivalent to the operator equation

$$\mathbf{v} = K(\mathbf{v}, c) = cA\mathbf{v} + R\mathbf{v} \tag{1.9}$$

where K is an operator completely continuous in H_1; cA is its Fréchet differential at the point $\mathbf{v} = 0$; A is independent of c .

Let us write the definition of the operators K and A . Let us fix some vector $\mathbf{v} \in H_1$ and let $T' \in H_2$ be a generalized solution of the equation

$$\chi \Delta T' - \mathbf{v} \cdot \nabla T' = f \tag{1.10}$$

with periodicity boundary conditions and (1.4). If $f \in L_{6/5}(\Omega)$, then as has been shown in [5], this generalized solution exists and is unique. The linear operator $B_{\mathbf{v}}$ has thereby been defined and $L_{6/5}(\Omega) \to H_2$

$$T' = B_{\mathbf{v}} f \tag{1.11}$$

We now obtain from the second equation of (1.3)

$$T = cB_{\mathbf{v}}\mathbf{v}_3 \equiv cM\mathbf{v} \tag{1.12}$$

Let us now determine the operator L which sets the vector $\mathbf{v} \in H_1$, the generalized solution of the linearized Navier-Stokes equations with right side $\mathbf{f}$:

$$\nu \Delta \mathbf{v} - \nabla p = \mathbf{f}, \qquad \operatorname{div} \mathbf{v} = 0, \qquad \mathbf{v} = L\mathbf{f} \tag{1.13}$$

with boundary conditions (1.4) and (1.5), in correspondence with the arbitrary vector $\mathbf{f} \in L_{6/5}(\Omega)$ (*) . It is now easy to see that

$$K(\mathbf{v}, c) = L(\mathbf{v}, \nabla)\mathbf{v} + cL(\beta \mathbf{g} B_{\mathbf{v}} v_3), \qquad A_{\mathbf{v}} = L(\beta \mathbf{g} B_0 v_3), \qquad B_0 = B_{\mathbf{v}}|_{\mathbf{v}=0}$$

$$R\mathbf{v} = L(\mathbf{v}, \nabla)\mathbf{v} + cL(\beta \mathbf{g}(B_v - B_0)v_3) \tag{1.14}$$

The operator A is self-adjoint, strictly positive, and its spectrum consistes of a sequence of positive eigenvalues.

Utilizing the principle of compression mappings, we obtain the following expression for the operator $B_{\mathbf{v}}$, which is valid for small $\mathbf{v} \in H_1$: (1.15)

$$B_{\mathbf{v}} f = B_0 f + B_1 f + \ldots + B_k f + \ldots, \qquad B_k f = B_0(\mathbf{v} \cdot \nabla B_{k-1} f) \ (k = 1, 2, \ldots)$$

3. Bifurcation . According to a theorem of Krasnosel'skii on bifurcation [7], every prime characteristic number of the Fréchet differential of A is a bifurcation point of (1.9). Since the operator A is self-adjoint, its characteristic number is prime if just one eigenvector corresponds to it.

It is verified directly that the operators K, A transform the space H_1° into itself. Later in this section, we shall consider the equation (1.9) in H_1°. From the definition (1.14) of the operator A it is implicit that the operator equation

$$\mathbf{v} = cA\mathbf{v} \tag{1.16}$$

is equivalent to the linearized system

$$\nu \Delta \mathbf{v} - \nabla p = \beta T \mathbf{g}, \qquad \chi \Delta T = c v_3, \qquad \operatorname{div} \mathbf{v} = 0 \tag{1.17}$$

with the boundary conditions (1.4) and (1.5). By a well-known method this latter is reduced to the single equation

---

*) If the problem for the layer is represented as the limit of appropriate problems for cylinders with unboundedly increasing radii, then the limiting form of the adhesion condition on the lateral surface of the cylinders is (1.5).

$$\Delta^3 T = \beta g c / \chi\nu \,(T_{x_1x_1} + T_{x_2x_2}) \tag{1.18}$$

with periodicity boundary conditions and

$$T = \Delta T = \Delta T_z = 0 \qquad (z = 0,\ h) \tag{1.19}$$

Because $\mathbf{v} \in H_1^2$, v_3 is an even function in x_1, x_2; then, according to (1.17), T is also even in x_1, x_2. Hence, T should be sought in the form

$$T = \tau(z) \cos n_1 k_1 x_1 \cos n_2 k_2 x_2 \tag{1.20}$$

in which the function $\tau(z)$ is a solution of the eigenvalue problem

$$-\left(\frac{d^2}{dz^2} - \theta^2\right)^3 \tau = \lambda\tau, \qquad \tau\,|_{z=0,\,h} = \tau'' - \theta^2\tau\,|_{z=0,\,h} = \tau''' - \theta^2\tau'\,|_{z=0,\,h} = 0$$
$$\theta^2 = (n_1k_1)^2 + (n_2k_2)^2, \qquad \lambda = \frac{\beta g \theta^2 c}{\chi\nu} \tag{1.21}$$

Here n_1, n_2 are natural numbers. Exactly as in [2], it can be shown that the Green's function of problem (1.21) is oscillatory (it is the composition of two symmetric oscillatory Green's functions). Hence, the spectrum of problem (1.9) is a sequence of positive and prime eigenvalues

$$\lambda_1(\theta) < \lambda_2(\theta) < \ldots < \lambda_n(\theta) < \ldots\ .$$

Thus, the spectrum of the problem (1.3) to (1.5) consists of the eigenvalues

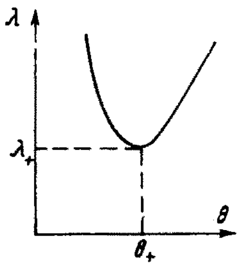

Fig. 1

$$c_{n_0n_1n_2} = \frac{\chi\nu}{\beta g}\frac{\lambda_n(\theta)}{\theta^2}, \quad \theta^2 = (n_1k_1)^2 + (n_2k_2)^2$$
$$(n_0,\ n_1,\ n_2 = 1,\ 2, \ldots) \tag{1.22}$$

If $c_{n_0n_1n_2}$ is a multiple eigen number, then natural numbers $(n_0',\ n_1',\ n_2') \neq (n_0,\ n_1,\ n_2)$, should also be found such that

$$c_{n_0n_1n_2} = c_{n_0'n_1'n_2'}, \quad \varphi(k_1, k_2) \equiv \frac{\lambda_n(\theta)}{\theta^2} - \frac{\lambda_{n'}(\theta')}{\theta'^2} = 0$$
$$\theta'^2 = (n_1'k_1)^2 + (n_2'k_2)^2 \tag{1.23}$$

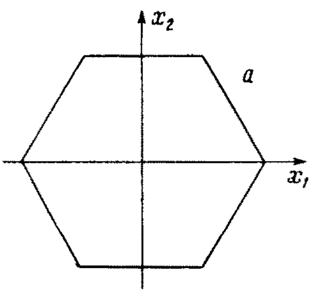

Fig. 2

Let us fix the arbitrary numbers n_i, n_i' $(i = 0, 1, 2)$. Exactly as in [2], it is shown that $\lambda_n(\theta)$ is an analytic function on the ray $\theta > 0$, and hence, the function $\varphi(k_1, k_2)$ is analytic within the quadrant $(k_1 > 0,\ k_2 > 0)$. As in [2], the function φ may not be identically zero. We thereby arrive at the following theorem.

T h e o r e m 1.1. For almost all pairs (k_1, k_2) each of the eigen numbers $c_{n_0n_1n_2}$ is prime, and this means it is a bifurcation point of (1.9): for values of c close to $c_{n_0n_1n_2}$ Equation (1.9), and so problem (1.3) and (1.5), have a nontrivial solution.

Here the expression "for almost all pairs (k_1, k_2)" is understood in the sense that on any analytic curve in the (k_1, k_2) plane there lies not more than a countable set of exclusive points.

As calculations show [3], the graph of the function $\lambda = \lambda_1(\theta)$ has the form indicated in Fig.1. It is hence clear that in every case λ_+ is a bifurcation point.

2. Cellular convection. It is well known and easily discerned from (1.22) that secondary flows with diverse periodicity and symmetry correspond to the very same critical value of the temperature gradient c. In this Section, secondary flows possessing hexagonal symmetry are examined. For such flows the layer is divided into regular hexagonal prisms, and the fluid motion in all such prisms proceeds identically. These flows (designated hexagonal or cellular convection) are of particular interest since precisely these are observed in experiments [6]; apparently cellular convection even occurs in the Earth's atmosphere, resulting in the formation of certain kinds of clouds.

Let a be the side of a hexagonal cell (Fig.2). Then the periods in x_1, x_2 should be $3a$, $a\sqrt{3}$, respectively. Moreover, the solution should be invariant relative to rotation through an angle $2\pi/3$ around the z-axis. This latter demand reduces to the conditions (2.1)

$$T(gx, z) = T(x, z), \quad v_3(gx, z) = v_3(x, z), \quad \mathbf{w}(gx, z) = g\mathbf{w}(x, z)$$

Here and henceforth, we use the notation: $x = (x_1, x_2)$; $\mathbf{w} = (v_1, v_2)$; g is the transformation of a rotation through the angle $2\pi/3$ around the z-axis (2.2)

$$g(x, z) = (gx, z) = (x_1 \cos {}^2/_3\pi - x_2 \sin {}^2/_3\pi, \ x_1 \sin {}^2/_3\pi + x_2 \cos {}^2/_3\pi, \ z)$$

We call the function $T(x, z)$ (the vector $\mathbf{v}(x, z)$, correspondingly) hexagonally symmetric if it satisfies condition (2.1) and is periodic in x_1, x_2 with the periods $3a$, $a\sqrt{3}$, respectively.

For the smooth function $T(x, z)$ to possess hexagonal symmetry it is necessary and sufficient that it be expanded in the series

$$T(x, z) = {}^1/_3 \sum c(n, z) \left[e^{ik(n,x)} + e^{ik(n,gx)} + e^{ik(n,g^{-1}x)}\right]$$
$$n = (n_1, n_2\sqrt{3}), \qquad k = {}^2/_3\pi/a, \qquad c(n, z) = c(gn, z) \tag{2.3}$$

Here the sum extends over all possible pairs of integers n_1, n_2 of the same evenness (this follows from the requirement that the lattice $(n_1, n_2\sqrt{3})$ be transformed into itself by the transformation g). For the proof it is enough to substitute a Fourier series of the function T into the equality

$$T(x, z) = {}^1/_3 \left[T(x, z) + T(gx, z) + T(g^{-1}x, z)\right] \tag{2.4}$$

which results from (2.1).

The coefficients $c(n, z)$ are determined uniquely by the function T. The Fourier expansion of the hexagonal vectors may be considered analogously.

Let us now introduce the subspaces $Q_1 \subset H_1^\circ$, $Q_2 \subset H_2^\circ$, consisting, respectively, of hexagonal vectors and functions.

For any function $T \in Q_2$ the expansion (2.3) becomes

$$T(x, z) = \Sigma a(n, z)\, \varphi_{n_1 n_2}(x), \qquad a(n, z) = a(gn, z) \tag{2.5}$$

$$\varphi_{n_1 n_2}(x) = \cos(kn_1x_1)\cos(k\sqrt{3}n_2x_2) + \cos\frac{k(n_1-n_2)\sqrt{3}\,x_2}{2} \times$$
$$\times \cos\frac{k(n_1+3n_2)\,x_1}{2} + \cos\frac{k(n_1-3n_2)\,x_1}{2}\cos\frac{k(n_1+n_2)\sqrt{3}\,x_2}{2}$$

The coefficients $a(n, z)$ are here real, and the summation is over all possible pairs of natural numbers of the same evenness. For $n_1 = n_2 = 1$ the function $\varphi_{n_1 n_2}(x)$ becomes

$$\varphi_{11}(x) = \cos[k(x_1 + x_2\sqrt{3})] + \cos[k(x_1 - x_2\sqrt{3})] + \cos(2kx_1) \tag{2.6}$$

which is customarily (although without sufficiently rigorous foundation) used to determine the sides of the hexagonal cell (see [3], for example).

Lemma 2.1. The operators K, A, R operate in the subspace Q_1 and the operator N from Q_1 in Q_2.

Proof. The transformation g generates an operator Γ_g operating on the function $f(x, z)$, and an operator Γ_g operating on the vector $\mathbf{v}(x, z)$ according to the rules

$$f_g = \Gamma_g f = f(gx, z), \qquad \mathbf{v}_g = \Gamma_g \mathbf{v} = \{g^{-1}\mathbf{w}(gx, z),\ \mathbf{v}_3(gx, z)\} \tag{2.7}$$

Condition (2.1) for the hexagonal symmetry of the function f or the vector $\mathbf{v}$ is now written as $f = f_g$, $\mathbf{v} = \mathbf{v}_g$. The relationships

$$\begin{gathered}\nabla f_g = \Gamma_g \nabla f, \qquad \Delta f_g = (\Delta f)_g, \qquad \Gamma_g(\mathbf{v}\cdot\nabla T) = \mathbf{v}_g \cdot \nabla T_g \\ \Gamma_g(\mathbf{v}, \nabla)\,\mathbf{v} = (\mathbf{v}_g, \nabla)\mathbf{v}_g, \qquad \Delta\mathbf{v}_g = (\Delta\mathbf{v})_g\end{gathered} \tag{2.8}$$

are verified directly.

Let us consider the operator N as an example. Let $\mathbf{v} \in Q_1$. Let us apply the operator Γ_g to (1.10) for $f = \mathbf{v}_3$. Utilizing the identities (2.8) and taking into account that $\mathbf{v}_g = \mathbf{v}$, we obtain

$$\chi\Delta T_g' - \mathbf{v}\cdot\nabla T_g' = \mathbf{v}_3 \tag{2.9}$$

Evidently the function T_g' satisfies the periodicity boundary conditions and (1.4). By virtue of the uniqueness of the solution (2.9) under these conditions, $T_g' = T'$; the evenness of the function T' in x_1, x_2 is evident Lemma is proved.

Let us now consider (1.9) in the space Q_1. As in Section 1, the corresponding linearized problem reduces to (1.17). Hexagonal solutions of this latter are

$$T_{n_1 n_2} = \tau(z)\,\varphi_{n_1 n_2}(x), \qquad \mathbf{v}_{n_1 n_2} = L(\beta T_{n_1 n_2}\mathbf{g}) \tag{2.10}$$

where n_1, n_2 are any natural numbers of the same evenness; L is the operator defined in (1.13), and the function $\tau(z)$ is the eigen solution of the problem (1.21) for $\theta^2 = (n_1^2 + 3n_2^2)k^2$.

Reasoning further, exactly as in the proof of Theorem 1.1, we obtain the following assertion.

Theorem 2.1. For any value of the side a of the hexagonal cell enclosing some countable set, Equation (1.16) in Q_1 has a sequence of positive prime eigenvalues, each of which is a bifurcation point of Equation (1.9); for values of the parameter c close to them Equation (1.9) (and the problem (1.13) as well) has small nonzero hexagonal solutions.

An analogous result may also be obtained for "triangular convection", which is a particular case of hexagonal convection.

BIBLIOGRAPHY

1. Iudovich, V.I., Primer rozhdeniia vtorichnogo statsionarnogo ili periodicheskogo techeniia pri potere ustoichivosti laminarnogo techeniia viazkoi neszhimaemoi zhidkosti (Example of the generation of the secondary stationary or periodic flow when there is loss of stability of the laminar flow of a viscous incompressible fluid). *PMM* Vol.29, № 3, 1965.
2. Iudovich, V.I., Vtorichnye techeniia i neustoichivost' zhidkosti mezhdu vrashchaiushchimisia tsilindrami (Secondary flows and fluid instability between rotating cylinders). *PMM* Vol.30, № 4, 1966.
3. Pellew, A. and Southwell, R.V., On maintained convective motion in a fluid heated from below. Proc.Roy.Soc., Ser.A, Vol.176, 1940.
4. Sorokin, V.S., O statsionarnykh dvizheniiakh v zhidkosti podogrevaemoi snizu (On stationary motions in a fluid heated from below). *PMM* Vol. 18, № 2, 1954.
5. Ukhovskii, M.R.,and Iudovich, V.I., Ob uravneniiakh statsionarnoi konvektsii (On the equations of steady-state convection). *PMM* Vol.27, № 2, 1963.
6. Bénard, M., Les tourbillons cellulaires dans une nappe liquide transportant de la chaleur par convection en régime permanent. Ann.Chim., ser.7, Vol.23, 1901.
7. Krasnosel'skii, M.A., Topologicheskie metody v teorii nelineinykh integral'nykh uravnenii (Topological Methods in Nonlinear Integral Equation Theory). Gostekhizdat, 1956.

Translated by M.D.F.